PERGAMON

International Journal of Solids and Structures 40 (2003) 5477–5494

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Free and forced vibration of repetitive structures

Dajun Wang [a], Chunyan Zhou [a], Jie Rong [b,*]

[a] Department of Mechanics and Engineering Science, Peking University, Beijing 100871, PR China

[b] Department of Aerospace Engineering, Texas A&M University, TAMU 3141, College Station, TX 77840-3141, USA

Received 19 April 2002; received in revised form 27 April 2003

Abstract

In this paper, the vibration problems of some repetitive structures, including symmetric, cyclic periodic, linear periodic, chain, and axi-symmetric structures is investigated. Eigen-value problems derived from the vibration equations of these structures are established based on their continuous models. The special properties of the structural modes of these structures are deduced. Applying these properties can provide effective reduction approach to solving the natural and forced vibration problems of these structures by either numerical or experimental methods. Furthermore, these properties can be applied in other aspects such as evaluating the reasonableness of the discrete models of these repetitive structures.

Keywords: Repetitive structures; Symmetric structures; Cyclic periodic structures; Linear periodic structures; Chain structures; Axi-symmetric structures; Vibration; Mode

1. Introduction

Repetitive structures are very common both in engineering and in natural world. A structure can be defined as repetitive when it is assembled in a proper way by a group of identical substructures. These substructures are identical in terms of geometric shape, physical properties, boundary conditions, and connections with other substructures.

Generally, the modes and natural frequencies of a structure are determined by the nature of the whole structure. However, for a repetitive structure, by virtue of its special repetitive property, the modes and natural frequencies of the whole structure can be determined from those of its single substructure. Thus, effort of calculating or measuring the free and forced vibration of these kinds of structures can be considerably reduced. Up to now, many papers have been discussing the dynamic properties of the repetitive structures: Evensen (1976) studied the vibration of symmetric structures; Thomas (1979) presented some results concerning the vibration of cyclic periodic structures; A series of work was done on analyzing the vibration of periodic structures using the U-transformation method (Cai and Wu, 1983; Cai et al., 1990;

* Corresponding author. Tel.: +1-979-220-2744.
E-mail address: jierong@tamu.edu (J. Rong).

0020-7683/$ - see front matter
doi:10.1016/S0020-7683(03)00279-8

Chan et al., 1998); Hu and Chen (1988) analyzed the vibration of cyclic periodic structures using C_N group method; Zhong (1991) gave several approaches for solving the eigen-value problems of linear periodic structures; Wang and Wang (2000) provided some simplification methods for solving the discrete eigen-value problems of specific kinds of repetitive structures. Results of these researches have been widely used in the design of many kinds of FEM software.

Some other researchers, including Brillouin (1946) and Mead (1973, 1975a,b) and Mead and Bansal (1978) have studied the wave propagations in linear periodic systems. Mead has done lots of work on the approximate solution of the propagating wave of linear periodic systems, using the "propagation constant". Since only the exact solution of standing waves are concerned in the present paper, the propagating wave motion is not discussed.

It is necessary to extend the existing research results of discrete models of repetitive structures to continuous models. In most current literatures concerning the vibration problems of repetitive structures, for the sake of simplifying calculation, discrete models of structures are studied and eigen-value problems of matrices with repetitive properties are solved. In this paper, continuous models of repetitive structures are studied. Although analyzing eigen-value problems of differential equations may be more difficult in mathematics, results of continuous models are more fundamental in nature. The derived qualitative properties of the structural modes (and frequencies) are of great importance in physics. They can be used to simplify the numerical calculation and the experiments of natural and forced vibration problems. In addition, they can be used to evaluate the correctness of the data obtained in the numerical calculation and the experiments of vibration problems, and to identify the reasonableness of discrete models of repetitive structures.

In this paper, the unique qualitative properties of natural and forced vibration of some repetitive structures, including symmetric, cyclic periodic, linear periodic, chain and axi-symmetric structures, are investigated. Each structure is analyzed in three steps: (1) Eigen-value problem of its differential vibration equation is established based on the continuous model of the structure; (2) The qualitative properties of its modes are deduced using the specific transformation on its displacement function field; (3) Application of the derived properties to simplifying the calculation and the experiment of natural and forced vibration problems is discussed, and some examples of application to the real structures are presented.

2. Symmetric structures

2.1. Model and equation

A structure is defined as mirror-symmetric or symmetric for short, if its geometric shape, physical properties as well as boundary conditions are all symmetric with respect to a plane (or a strait line) that is called as a symmetric plane (or a symmetric line).

An example of a symmetric structure is illustrated in Fig. 1. Plane $x = 0$ in a Cartesian coordinate system is the symmetric plane. It divides the whole structure into two substructures, no. 1 and no. 2, both of which are identical in shape, physical properties, and boundary conditions. Two Cartesian coordinate systems are set respectively in substructures no. 1 and no. 2, the directions of their y and z axes are the same, while those of their x axes are opposite to each other. The generalized displacement vectors of the two substructures are denoted by $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ respectively, and the generalized displacement vector of the whole structure is $\boldsymbol{w} = (\boldsymbol{w}_1, \boldsymbol{w}_2)^{\mathrm{T}}$.

The eigen-value equation and boundary conditions are expressed as follows,

$$\boldsymbol{L}\boldsymbol{w}_i - \omega^2 \boldsymbol{M}\boldsymbol{w}_i = 0 \quad \text{in } \boldsymbol{\Omega} \quad i = 1, 2 \tag{1}$$

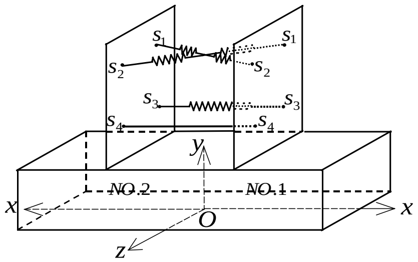

Fig. 1. A symmetric structure.

$$\boldsymbol{B}\boldsymbol{w}_i = 0 \quad \text{on } \partial\boldsymbol{\Omega} \quad i = 1,2 \tag{2}$$

where $\boldsymbol{L}$, $\boldsymbol{M}$, and $\boldsymbol{B}$ are elastic, mass, and boundary differential operators or differential operator matrices respectively. $\boldsymbol{\Omega}$ is the region of a substructure and $\partial\boldsymbol{\Omega}$ is its boundary excluding the common boundary where x is equal to 0.

It should be noted that on the common boundary of substructure no. 1 and no. 2, their generalized displacements and generalized internal forces satisfy the continuous conditions, which are expressed in terms of differential equations as follows,

$$\boldsymbol{J}_1\boldsymbol{w}_1 = -\boldsymbol{J}_1\boldsymbol{w}_2 \quad \text{on } x = 0 \tag{3}$$

$$\boldsymbol{J}_2\boldsymbol{w}_1 = \boldsymbol{J}_2\boldsymbol{w}_2 \quad \text{on } x = 0 \tag{4}$$

As shown in Fig. 1, the lower part of the structure is a three-dimensional elastic body and the upper part is composed of two rectangular plates. On their common boundary (plane $x = 0$) in the elastic body, the continuous conditions of the displacements and stresses of substructure no. 1 and no. 2 are

$$\begin{bmatrix} 1 & 0 & 0 \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix} = - \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \\ \dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dfrac{\partial}{\partial x} & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dfrac{\partial}{\partial x} & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

The elastic and rigid constraints between substructure no. 1 and no. 2, if exist, are written as follows:

$$\boldsymbol{J}_{rj}\boldsymbol{w}_1|_{s_j} = \overline{\boldsymbol{J}}_{rj}\boldsymbol{w}_2|_{\bar{s}_j} \quad j = 1,2,\ldots,l \tag{5}$$

$$\boldsymbol{J}_{rj}\boldsymbol{w}_2|_{s_j} = \overline{\boldsymbol{J}}_{rj}\boldsymbol{w}_1|_{\bar{s}_j} \quad j = 1,2,\ldots,l \tag{6}$$

For example, for the structure shown in Fig. 1, there are three springs and one rigid rod connecting the two plates, which means $l = 4$ and Eq. (5) is written as

$$Q_1(s_1) + k_1 \sin^2 \alpha u_1(s_1) = -k_1 \sin^2 \alpha u_2(s_2)$$

$$Q_1(s_2) + k_1 \sin^2 \alpha u_1(s_2) = -k_1 \sin^2 \alpha u_2(s_1)$$

$$Q_1(s_3) + k_3u_1(s_3) = -k_3u_2(s_3)$$

$$u_1(s_4) = -u_2(s_4)$$

where, $Q_1(s_i)$ corresponds to the spring forces acting at point s_i on the plates of substructures no. 1, and α represents the angle between the spring and the plate. Eq. (6) can be expressed in a similar way. Due to the continuous conditions and the constraints, $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are coupled. The vibration equations for the whole structure are expressed in Eqs. (1)–(6).

2.2. Simplification of eigen-value problem and qualitative properties of modes

The original displacements $\{w_1, w_2\}$ can be transferred into another set of generalized displacements $\{q_1, q_2\}$ as follows:

$$\boldsymbol{w} = \begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{Bmatrix} = \boldsymbol{S}\boldsymbol{q} = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix}\begin{Bmatrix} \boldsymbol{q}_1 \\ \boldsymbol{q}_2 \end{Bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{I} \end{bmatrix}\boldsymbol{q}_1 + \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix}\boldsymbol{q}_2 \tag{7}$$

where $\boldsymbol{I}$ is a unit matrix with the same dimensions of the displacement function field $\boldsymbol{w}_1$, the first and the second terms on the right side of the last equal-sign correspond to symmetric and anti-symmetric modes respectively. The transformation matrix is an orthogonal one, which means

$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{S} = \boldsymbol{I} \tag{8}$$

In Eq. (8), $\boldsymbol{I}$ is a unit matrix whose dimension is double of that of $\boldsymbol{w}_1$. Rewrite the Eqs. (1)–(6) as follows:

$$\begin{bmatrix} \boldsymbol{L} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{L} \end{bmatrix}\begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{Bmatrix} - \omega^2\begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} \end{bmatrix}\begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{Bmatrix} = \boldsymbol{0} \quad \text{in } \boldsymbol{\Omega}' \tag{9a}$$

$$\begin{bmatrix} \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{bmatrix}\begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{Bmatrix} = \boldsymbol{0} \quad \text{on } \partial\boldsymbol{\Omega}' \tag{9b}$$

$$\begin{bmatrix} \boldsymbol{J}_1 & \boldsymbol{J}_1 \\ \boldsymbol{J}_2 & -\boldsymbol{J}_2 \end{bmatrix}\begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{Bmatrix} = \boldsymbol{0} \quad \text{on } x = 0 \tag{9c}$$

$$\begin{bmatrix} \boldsymbol{J}_{rj} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_{rj} \end{bmatrix}\begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{Bmatrix}\Bigg|_{s_j} = \begin{bmatrix} \overline{\boldsymbol{J}}_{rj} & \boldsymbol{0} \\ \boldsymbol{0} & \overline{\boldsymbol{J}}_{rj} \end{bmatrix}\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0} \end{bmatrix}\begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \end{Bmatrix}\Bigg|_{\bar{s}_j} \quad i = 1, 2, \ldots, p \tag{9d}$$

where $\boldsymbol{\Omega}'$ and $\partial\boldsymbol{\Omega}'$ are the region and the boundary of the whole structure. Substituting Eq. (7) into Eqs. (9), then pre-multiplying (9a), (9b) and (9d) with S^{T}, the uncoupled equations of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ are derived as follows:

$$\begin{aligned} &\boldsymbol{L}\boldsymbol{q}_i - \omega^2\boldsymbol{M}\boldsymbol{q}_i = \boldsymbol{0} \quad \text{in } \boldsymbol{\Omega} \\ &\boldsymbol{B}\boldsymbol{q}_i = \boldsymbol{0} \quad \text{on } \partial\boldsymbol{\Omega} \\ &\boldsymbol{J}_i\boldsymbol{q}_i = \boldsymbol{0} \quad \text{on } x = 0 \;\; i = 1, 2 \\ &\boldsymbol{J}_{rj}\boldsymbol{q}_i|_{s_j} = \pm\overline{\boldsymbol{J}}_{rj}\boldsymbol{q}_i|_{\bar{s}_j} \quad (+ \text{ if } i = 1, - \text{ if } i = 2) \;\; j = 1, 2, \ldots, l \end{aligned} \tag{10}$$

We have two conclusions regarding to the symmetric structure: (1) The eigen-value problem of the whole structure expressed in Eqs. (9) can be simplified into two eigen-value problems of a single substructure expressed in Eqs. (10). Eq. (7) indicates that the solution of Eqs. (10) is the symmetric mode of the whole structure for $i = 1$, and it is the anti-symmetric mode for $i = 2$; (2) The mode of a symmetric structure is

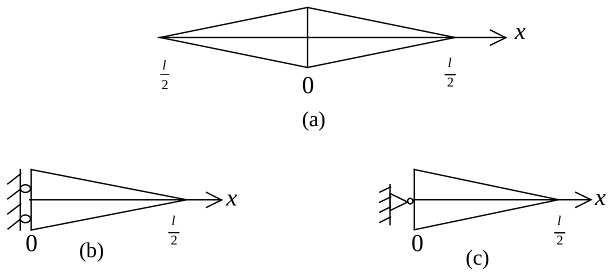

Fig. 2. Reduction of rhombus beam (a) original symmetric rhombus beam, (b) equivalent substructure for symmetric mode and (c) equivalent substructure for anti-symmetric mode.

either symmetric, or anti-symmetric, or the linear combination of a symmetric mode and an anti-symmetric mode with the same natural frequency.

2.3. Application

(1) When calculating the modes and natural frequencies of a symmetric structure, we only need to do calculation of one half of the structure. First, proper constraints and boundary conditions, representing symmetric or anti-symmetric deformation of the whole structure, need to be given on the symmetric plane. Then two eigen-value problems of one of the substructures are solved separately. The obtained frequencies of one substructure are exactly those of the whole structure. The modes of the whole structure can be obtained by expanding symmetrically or anti-symmetrically the obtained modes of one substructure. The advantage of this method is that the DOF required for the computation may be reduced by one half.

If we want to obtain the modes and natural frequencies of a symmetric structure by experiment, we only need to do measurement on one half of the whole structure and at one point on the other half. The modes of the whole structure can be expanded from the modes of its one substructure, either symmetrically or anti-symmetrically, according to whether the obtained data at two symmetric points on the structure are symmetric or anti-symmetric. However, if the data at two symmetric points indicates that the mode is neither symmetric nor anti-symmetric, we can claim that it implies a repeated frequency. We may get the symmetric and the anti-symmetric mode corresponding to the same frequency by making some adjustment in the experiment.

2.4. Examples

The Rhombus beam shown in Fig. 2 has a free–free boundary condition. It can be simplified as a sliding-free beam or a pinned-free beam, both of which have analytic solutions (Kirchhoff, 1879) for their eigen-value equations.

3. Cyclic periodic structures

3.1. Model and equation

A structure can be termed cyclic periodic if it is in form of an assembly of identical substructures that are distributed evenly on a circular ring. Once the geometric shape, physical properties, boundary conditions and its mutual connections with other substructures of one substructure are defined, those of the remainder

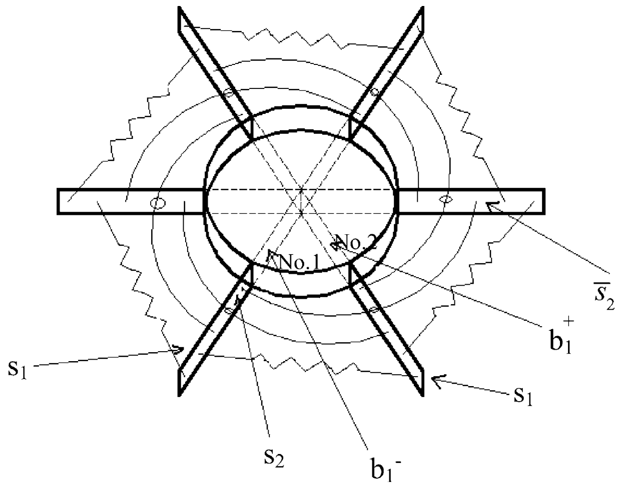

Fig. 3. A cyclic periodic structure.

of the whole structure can been obtained by rotating the structure repeatedly by angle $\psi = 2\pi/n$, where n represents the number of substructures. An example of a cyclic periodic structure is illustrated in Fig. 3.

Denote the kth substructure of a cyclic periodic structure by $\boldsymbol{\Omega}$, two common boundaries connecting kth substructure with the $(k-1)$th and the $(k+1)$th substructure respectively by b_k^- and b_k^+, and other boundaries by $\partial\boldsymbol{\Omega}$. The eigen-value problem of the differential equation for the whole structure is as follows:

$$\boldsymbol{L}\boldsymbol{w}_k - \omega^2 \boldsymbol{M}\boldsymbol{w}_k = 0 \quad k = 1, 2, \ldots, n \quad \text{in } \boldsymbol{\Omega} \tag{11}$$

$$\boldsymbol{B}\boldsymbol{w}_k = 0 \quad k = 1, 2, \ldots, n \quad \text{on } \partial\boldsymbol{\Omega} \tag{12}$$

where $\boldsymbol{w}_k$ represents the mode on the kth substructure in terms of function or function vector. $\boldsymbol{L}$, $\boldsymbol{M}$ and $\boldsymbol{B}$ represent the elastic, mass and boundary conditions operators or operator matrices of a substructure.

If common boundaries exist between two adjacent substructures, generalized displacements and generalized internal forces on the common boundary are continuous,

$$\boldsymbol{J}_0 \boldsymbol{w}_k|_{b_k^+} = \boldsymbol{J}_0 \boldsymbol{w}_{k+1}|_{b_{k+1}^-} \quad k = 1, 2, \ldots, n \tag{13}$$

where $\boldsymbol{w}_{n+1} \equiv \boldsymbol{w}_1$, b_{n+1}^- is b_1^-, $\boldsymbol{J}_0$ represents a differential operator or differential operator matrix.

If the elastic and rigid constraints between two substructures exist, they are expressed as follows:

$$\boldsymbol{J}_{pj} \boldsymbol{w}_k|_{s_{pj}} = \overline{\boldsymbol{J}}_{pj} \boldsymbol{w}_{k+p}|_{\overline{s}_{pj}} \quad k = 1, 2, \ldots, n \;\; p = 1, 2, \ldots, n-1 \;\; j = 1, 2, \ldots, l_p \tag{14}$$

where $\boldsymbol{J}_{pj}$ $(p = 1, 2, \ldots, n-1)$ denotes a differential operator or differential operator vector. The subscript of $\boldsymbol{w}_{k+p}$ is set as i if it reaches to $n+i$. The pth equation indicates the constraints between the region s_{pj} (a point or a region of one to three dimensions) in the kth substructure and the region $\overline{s}_{pj}$ in the $(k+p)$th substructure. When no constraint exists between the kth substructure and some other substructures, the corresponding equations in (14) will not appear.

For example, for a structure shown in Fig. 3, a spring is linking the point s_1 in the kth substructure with the point $\overline{s}_1$ in the $(k+1)$th substructure. Therefore, the first equation of (14) indicates a spring force acting at s_1 induced by relative displacement of point s_1 to $\overline{s}_1$. The point s_2 in the kth substructure is rigidly connected with the point $\overline{s}_2$ in the $(k+2)$th substructure, which in the second equation of (14) the displacements of point s_2 and those of point $\overline{s}_2$ should be equal. This example illustrates the case when $p = 1, 2$, $l_1 = l_2 = 1$.

Due to the continuous conditions (13) and the constraints (14) between two substructures, $\boldsymbol{w}_k$ $(k = 1, 2, \ldots, n)$ are coupled with each other, we need to solve the coupled equations of $\boldsymbol{w}_1$ to $\boldsymbol{w}_k$ if we try to calculate the natural frequencies and modes of the whole structure using Eqs. (11)–(14) directly.

3.2. Simplification of eigen-value problem and qualitative properties of modes

The original displacement $\boldsymbol{w}$ can be transferred into another set of generalized displacements as follows:

$$\begin{aligned}
&\boldsymbol{w} = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}^{\mathrm{T}} = [\,\boldsymbol{R}_1 \quad \boldsymbol{R}_2 \quad \cdots \quad \boldsymbol{R}_n\,]\{\,\boldsymbol{q}_1 \quad \boldsymbol{q}_2 \quad \cdots \quad \boldsymbol{q}_n\,\}^{\mathrm{T}} = \boldsymbol{R}\boldsymbol{q}, \\
&\boldsymbol{R}_r = \frac{1}{\sqrt{n}}[\boldsymbol{I}, \mathrm{e}^{\mathrm{i}r\psi}\boldsymbol{I}, \ldots, \mathrm{e}^{\mathrm{i}r(n-1)\psi}\boldsymbol{I}]^{\mathrm{T}}
\end{aligned} \tag{15}$$

where matrix $\boldsymbol{R}$ is a $\boldsymbol{U}$ matrix, i.e.,

$$\overline{\boldsymbol{R}}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{I} \tag{16}$$

The eigen-value equation of the whole structure, Eqs. (11)–(14), are rewritten as follows,

$$\boldsymbol{L}'\boldsymbol{w} - \omega^2\boldsymbol{M}'\boldsymbol{w} = 0 \quad \text{in } \boldsymbol{\Omega}' \tag{17}$$

$$\boldsymbol{B}'\boldsymbol{w} = 0 \quad \text{on } \partial\boldsymbol{\Omega}' \tag{18}$$

$$\boldsymbol{J}_0'\boldsymbol{w}|_{b^+} = \boldsymbol{J}_0'\boldsymbol{Y}\boldsymbol{w}|_{b^-} \tag{19}$$

$$\boldsymbol{J}_{pj}'\boldsymbol{w}|_{s_{pj}} = \overline{\boldsymbol{J}'}_{pj}\boldsymbol{Y}^p\boldsymbol{w}|_{\bar{s}_{pj}} \quad p = 1, 2, \ldots, n-1 \;\; j = 1, 2, \ldots, l_p \tag{20}$$

where $\boldsymbol{\Omega}'$ and $\partial\boldsymbol{\Omega}'$ are the region and the boundary of the whole structure, $\boldsymbol{L}'$, $\boldsymbol{M}'$, $\boldsymbol{B}'$, $\boldsymbol{J}_0'$, and $\boldsymbol{J}_{pj}'$, $\overline{\boldsymbol{J}'}_{pj}$ $(p = 1, 2, \ldots, n-1)$ represent the block diagonal matrices of $\boldsymbol{L}$, $\boldsymbol{M}$, $\boldsymbol{B}$, $\boldsymbol{J}_0$, and $\boldsymbol{J}_{pj}$, $\overline{\boldsymbol{J}}_{pj}$ respectively. Moreover,

$$\boldsymbol{Y} = \begin{bmatrix} 0 & \boldsymbol{I} & & & & & \\ & 0 & \boldsymbol{I} & & & & \\ & & \bullet & \bullet & & & \\ & & & \bullet & \bullet & & \\ & & & & \bullet & \bullet & \\ & & & & & 0 & \boldsymbol{I} \\ \boldsymbol{I} & & & & & & 0 \end{bmatrix} \quad \boldsymbol{Y}^p = \begin{bmatrix} 0 & \bullet & \overset{p+1}{0} & \boldsymbol{I} & 0 & \bullet & 0 \\ 0 & \bullet & 0 & 0 & \boldsymbol{I} & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet & \bullet & \bullet & \boldsymbol{I} \\ \boldsymbol{I} & 0 & 0 & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \underset{p}{\boldsymbol{I}} & 0 & 0 & \bullet & 0 \end{bmatrix} \tag{21}$$

where $\boldsymbol{Y}^p$ is termed row-switch transform matrix and

$$\overline{\boldsymbol{R}}^{\mathrm{T}}\boldsymbol{Y}^p\boldsymbol{R} = \mathrm{diag}(\,\mathrm{e}^{\mathrm{i}p\psi}\boldsymbol{I} \quad \mathrm{e}^{\mathrm{i}2p\psi}\boldsymbol{I} \quad \cdots \quad \mathrm{e}^{\mathrm{i}np\psi}\boldsymbol{I}) \tag{22}$$

Substituting Eq. (15) into Eqs. (17)–(20), then pre-multiplying with $\overline{\boldsymbol{R}}^{\mathrm{T}}$, and applying Eq. (16) and (22), we may obtain

$$\begin{aligned}
&\boldsymbol{L}\boldsymbol{q}_r - \omega^2\boldsymbol{M}\boldsymbol{q}_r = 0 \quad \text{in } \boldsymbol{\Omega} \\
&\boldsymbol{B}\boldsymbol{q}_r = 0 \quad \text{on } \partial\boldsymbol{\Omega} \\
&\boldsymbol{J}_0\boldsymbol{q}_r|_{b^+} = \boldsymbol{J}_0\,\mathrm{e}^{\mathrm{i}r\psi}\boldsymbol{q}_r|_{b^-} \\
&\boldsymbol{J}_{pj}\boldsymbol{q}_r|_{s_{pj}} = \overline{\boldsymbol{J}}_{pj}\,\mathrm{e}^{\mathrm{i}pr\psi}\boldsymbol{q}_r|_{\bar{s}_{pj}} \quad r = 1, 2, \ldots, n \;\; p = 1, 2, \ldots, n-1, \;\; j = 1, 2, \ldots, l_p
\end{aligned} \tag{23}$$

In Eqs. (23), $\boldsymbol{q}_r = \boldsymbol{q}_r^r + \mathrm{i}\boldsymbol{q}_r^i$ $(r = 1, 2, \ldots, n)$ are uncoupled. It can be verified that the complex solution of Eqs. (23) corresponding to $r = n - s$ is conjugated with that corresponding to $r = s$.

As to a cyclic periodic structure, we come to two conclusions: First, for a cyclic periodic structure, the eigen-value problem of the whole structure expressed in Eqs. (17)–(20) can be simplified into n eigen-value problems of one single substructure expressed in Eq. (23). By utilizing Eq. (15), the mode of the whole structure can be obtained from $\boldsymbol{w}^{(r)} = \boldsymbol{u}^{(r)} + \mathrm{i}\boldsymbol{v}^{(r)}$.

$$\begin{Bmatrix} \boldsymbol{u}^{(r)} \\ \boldsymbol{v}^{(r)} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{u}_1^r \\ \boldsymbol{u}_2^r \\ \vdots \\ \boldsymbol{u}_n^r \\ \boldsymbol{v}_1^r \\ \boldsymbol{v}_2^r \\ \vdots \\ \boldsymbol{v}_n^r \end{Bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \cos r\psi \boldsymbol{I} & -\sin r\psi \boldsymbol{I} \\ \vdots & \vdots \\ \cos r(n-1)\psi \boldsymbol{I} & -\sin r(n-1)\psi \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{I} \\ \sin r\psi \boldsymbol{I} & \cos r\psi \boldsymbol{I} \\ \vdots & \vdots \\ \sin(n-1)r\psi \boldsymbol{I} & \cos(n-1)r\psi \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{q}_r^r \\ \boldsymbol{q}_r^i \end{Bmatrix} \quad r = 1, 2, \ldots, n \tag{24}$$

Second, the modes of a cyclic periodic structure can be divided into n groups expressed as (24). For each group of modes, a specific phase lag exists between two adjacent substructures,

$$\boldsymbol{w}_{k+1}^{(r)} = \mathrm{e}^{\mathrm{i}r\psi} \boldsymbol{w}_k^{(r)} \tag{25}$$

These modes can be further categorized into three following classes,

(1) The displacements of every substructure are identical, which indicates that in Eq. (24), $r = n$, i.e.

$$\boldsymbol{w}^{(n)} = \{\boldsymbol{q}_n, \boldsymbol{q}_n, \ldots, \boldsymbol{q}_n\}^{\mathrm{T}} \tag{26}$$

If n is even, in Eq. (24) $r = n/2$, and the displacements of two adjacent substructures are opposite, i.e.

$$\boldsymbol{w}^{(n/2)} = \{\boldsymbol{q}_{n/2}, -\boldsymbol{q}_{n/2}, \ldots, -\boldsymbol{q}_{n/2}\}^{\mathrm{T}} \tag{27}$$

In any other case when $r \neq n$, $n/2$ (even n), there exist two modes associated with one repeated frequency

$$\boldsymbol{u}_1^{(r)}, \boldsymbol{u}_2^{(r)}, \ldots, \boldsymbol{u}_n^{(r)} \quad \text{and} \quad \boldsymbol{v}_1^{(r)}, \boldsymbol{v}_2^{(r)}, \ldots, \boldsymbol{v}_n^{(r)} \quad r = 1, 2, \ldots, \frac{n-2}{2} (\text{even } n) \text{ or } \frac{n-1}{2} \ (\text{odd } n)$$

and the relationship between them is

$$\begin{aligned} \boldsymbol{u}_{k+1}^{(r)} &= \cos r\psi \boldsymbol{u}_k^{(r)} - \sin r\psi \boldsymbol{v}_k^{(r)} \\ \boldsymbol{v}_{k+1}^{(r)} &= \sin r\psi \boldsymbol{u}_k^{(r)} + \cos r\psi \boldsymbol{v}_k^{(r)} \end{aligned} \tag{28}$$

3.3. Application

The process of calculating the frequencies and modes of a cyclic periodic structure can be divided into two steps. First, the real eigen-value equations of coupled $\boldsymbol{q}_r^r$ and $\boldsymbol{q}_r^i$ are solved as follows,

$$\boldsymbol{L}\boldsymbol{q}_r^r - \omega^2 \boldsymbol{M}\boldsymbol{q}_r^r = 0 \quad \boldsymbol{L}\boldsymbol{q}_r^i - \omega^2 \boldsymbol{M}\boldsymbol{q}_r^i = 0 \quad \text{in } \boldsymbol{\Omega} \tag{29}$$

$$\boldsymbol{B}\boldsymbol{q}_r^r = 0 \quad \boldsymbol{B}\boldsymbol{q}_r^i = 0 \quad \text{on } \partial\boldsymbol{\Omega} \tag{30}$$

$$\begin{aligned} \boldsymbol{J}_0 \boldsymbol{q}_r^r|_{b^+} &= \boldsymbol{J}_0(\cos r\psi \boldsymbol{q}_r^r - \sin r\psi \boldsymbol{q}_r^i)|_{b^-} \\ \boldsymbol{J}_0 \boldsymbol{q}_r^i|_{b^+} &= \boldsymbol{J}_0(\sin r\psi \boldsymbol{q}_r^r + \cos r\psi \boldsymbol{q}_r^i)|_{b^-} \end{aligned} \tag{31}$$

$$
\begin{aligned}
&\boldsymbol{J}_{pj}\boldsymbol{q}_r^r|_{s_{pj}} = \overline{\boldsymbol{J}}_{pj}(\cos rp\psi \boldsymbol{q}_r^r - \sin rp\psi \boldsymbol{q}_r^i)|_{\bar{s}_{pj}} \\
&\boldsymbol{J}_{pj}\boldsymbol{q}_r^i|_{s_{pj}} = \overline{\boldsymbol{J}}_{pj}(\sin rp\psi \boldsymbol{q}_r^r + \cos rp\psi \boldsymbol{q}_r^i)|_{\bar{s}_{pj}} \\
&p = 1,2,\ldots,n-1 \quad j = 1,2,\ldots,l_p, \quad r = 1,2,\ldots,\frac{n-2}{2} \text{ (if } n \text{ is even) or } \frac{n-1}{2} \text{ (if } n \text{ is odd)}
\end{aligned}
\tag{32}
$$

Second, according to Eq. (24), the modes of the whole structure can be derived from $\boldsymbol{q}_r^r$ and $\boldsymbol{q}_r^i$. It should be noted that when $r = n$ or $r = n/2$ (if n is even), equation (23) and its solution are real. Therefore, we only need to solve the eigen-value equations of $\boldsymbol{q}_n^r$ and $\boldsymbol{q}_{n/2}^r$ on a single substructure.

(1) If we suppose that each substructure of a discrete cyclic periodic system has m DOF, the eigen-value problem of the whole structure expressed in Eqs. (11)–(14) will have $n \times m$ DOF. However, for the uncoupled eigen-value problem expressed in Eqs. (29)–(32), we only need to solve $(n-2)/2$ (if n is even) or $(n-1)/2$ (if n is odd) eigen-value problems with $2 \times m$ DOF and two (n is even) or one (n is odd) eigen-value problem with m DOF. Therefore, the computational complexity is considerably reduced.

(2) If we want to obtain the modes and natural frequencies of a cyclic periodic system by experiment, we can follow two steps: (1) Measure only the mode $\boldsymbol{q}$ on a single substructure. (2) Select a point s on this substructure where $\boldsymbol{q}(s)$ is not equal to zero and measure $\boldsymbol{q}(s)$ at the same point on its adjacent substructure. If the two $\boldsymbol{q}(s)$ are identical, it indicates that the mode of the whole structure is in the form of $\boldsymbol{w}^{(n)} = [\boldsymbol{I} \quad \boldsymbol{I} \quad \cdots \quad \boldsymbol{I}]^{\mathrm{T}}\boldsymbol{q}$. If the two $\boldsymbol{q}(s)$ are opposite, the mode of the whole structure is in the form of $\boldsymbol{w}^{(n/2)} = [\boldsymbol{I} \quad -\boldsymbol{I} \quad \cdots \quad -\boldsymbol{I}]^{\mathrm{T}}\boldsymbol{q}$. If two modes, $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, on a single substructure are detected to be associated with a same natural frequency and at a same point s of two adjacent substructures, the following relations exist,

$$\boldsymbol{q}_{1,k+1}(s) = \cos r\psi \boldsymbol{q}_{1,k}(s) - \sin r\psi \boldsymbol{q}_{2,k}(s)$$

$$\boldsymbol{q}_{2,k+1}(s) = \sin r\psi \boldsymbol{q}_{1,k}(s) + \cos r\psi \boldsymbol{q}_{2,k}(s)$$

and thus modes $\boldsymbol{u}_r$ and $\boldsymbol{v}_r$ represented by exp.(24) are two modes with repeated frequencies.

3.4. Examples

As shown in Fig. 4, the plane frame is composed of four uniform beams rigidly connected with each other. The length of each beam is l and the four connection corners are pinned supported. Two kinds of

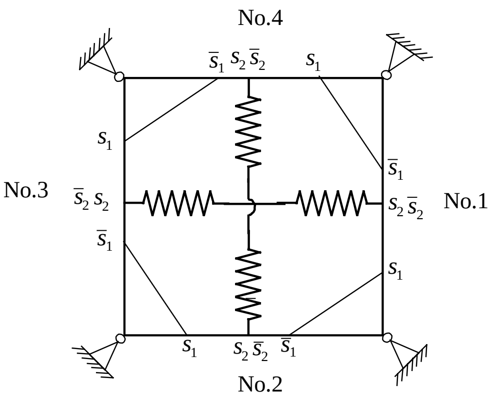

Fig. 4. A plane frame assembled by four beams.

constraints exist in this structure: (1) The transverse displacement of the point s_1 on the kth beam should be identical with that of the point $\bar{s}_1$ on the $(k+1)$th beam; (2) A spring, with the spring constant k, connects the middle point of the kth beam with that of the $(k+2)$th beam. The transverse displacement of the beam is denoted by w, while w' represents its differential term with respect to the coordinate parallel to the axis of the beam.

The continuous conditions and constraints are expressed as following:

$$w_k(l) = w_{k+1}(0) = 0$$

$$w'_k(l) = w'_{k+1}(0) \quad w''_k(l) = w''_{k+1}(0)$$

$$w_k(s_1) = w_{k+1}(\bar{s}_1)$$

$$Q_k\left(\frac{l}{2}\right) + Kw_k\left(\frac{l}{2}\right) = -Kw_{k+2}\left(\frac{l}{2}\right)$$

The modes and frequencies of this structure can be divided into three groups:

(1) When $r = 4$, q_4 denotes the mode of the beam shown in Fig. 5(a).

$$q'_4(0) = q'_4(l), \quad q''_4(0) = q''_4(l)$$

$$q_4(s_1) = q_4(\bar{s}_1)$$

$$Q_4\left(\frac{l}{2}\right) = -2Kq_4\left(\frac{l}{2}\right)$$

The mode of the whole structure is

$$\boldsymbol{w} = \{1 \quad 1 \quad 1 \quad 1\}^{\mathrm{T}} q_4(x)$$

(2) When $r = 2$, q_2 denotes the mode of the beam shown in Fig. 5(b).

$$q'_2(0) = -q'_2(l), \quad q''_2(0) = -q''_2(l)$$

$$q_2(s_1) = -q_2(\bar{s}_1), \quad Q\left(\frac{l}{2}\right) = -2kq_2\left(\frac{l}{2}\right)$$

The mode of the whole structure is

$$\boldsymbol{w} = \{1 \quad -1 \quad 1 \quad -1\}^{\mathrm{T}} q_2(x)$$

(3) When $r = 1$, q_1 denotes to the mode of the beam shown in Fig. 5(c).

$$q'_1(l) = \mathrm{i}q'_1(0), \quad q''_1(l) = \mathrm{i}q''_1(0)$$

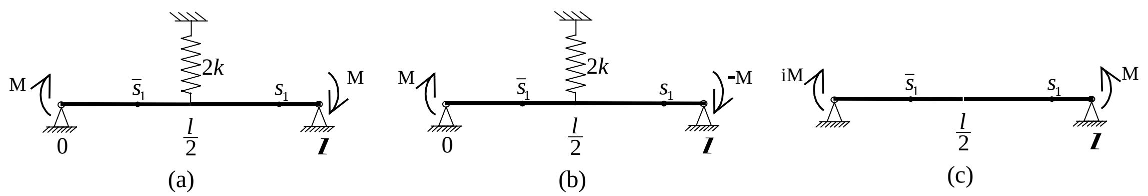

Fig. 5. Equivalent structures.

$$q_1(s_1) = \mathrm{i}q_1(\bar{s}_1)$$

The mode of the whole structure is

$$\boldsymbol{w} = \{1 \quad \mathrm{i} \quad -1 \quad -\mathrm{i}\}^{\mathrm{T}} q_1(x)$$

This problem is a complex eigen-vector problem associated with real eigen-values. It can also be expressed as real eigen-vector problems with real and imaginary parts coupled with each other.

4. Linear periodic structures

A linear periodic structure is composed of an assembly of identical substructures distributed evenly on a straight line (or a circular arc). All the substructure are identical in terms of the geometric shape, physical properties, boundary conditions and the constraints with other substructures, expect for the two substructures at the ends that can have peculiar boundary conditions.

For some special kinds of linear periodic structures, their eigen-value problems can be solved by utilizing the method for cyclic periodic structures. The calculation process is divided into two steps: (1) Extend the original structure by one or two times; (2) Generate a cyclic periodic structure by joining the two ends of the extended structure. The structure suitable for this method should satisfy two following pre-requisites:

All of its substructure should be symmetric, which means its geometric shape, physical properties, boundary conditions, and constraints with other substructures are all symmetric. Therefore, the newly generated cyclic periodic structure is also symmetric.

At the two ends of the original linear structure, the boundary conditions should conform to the symmetric or anti-symmetric modes restrictions on the corresponding symmetric planes of the newly generated cyclic periodic structure.

5. Chain structures

5.1. Model and equation

A chain structure is a special type of a linear periodic structure. A structure is called a chain structure when it is in the form of an assembly of identical substructures distributed evenly on a straight line (or a circular arc), and satisfies the following three restrictions: (1) Between any of the two substructures, there is no common boundary but elastic or rigid constraints without mass; (2) The constraints between one substructure and its preceding one should be identical to that between it and its following one; (3) The two ends of the structure should be fixed. These three restrictions make the chain structure unique compared with ordinary linear periodic structures. The spring–mass system illustrated in Fig. 6 is a typical example of a chain structure.

Another example of a chain structure is illustrated in Fig. 7. For each substructure, a spring connects the point s_1 on it with the points s_2 on its neighbors, another spring links the point s_2 on it with the points s_1 on

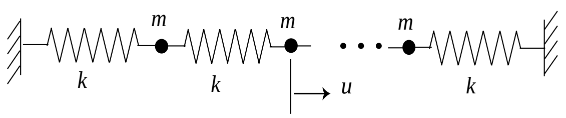

Fig. 6. Spring–mass system.

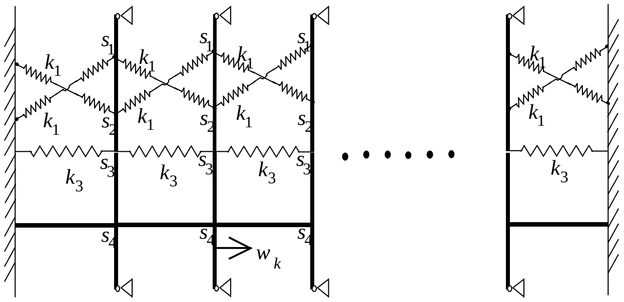

Fig. 7. Another chain structure.

its neighbors, the third spring connects point s_3 with the same point on its neighbors, and point s_4 is rigidly connected with the same point on its neighbors.

As a special kind of linear structure, the eigen-value equation of a chain structure can be solved by the general method mentioned in the preceding section for linear structures. However, by virtue of its distinguished feature, a more simple approach can be used.

If a chain structure is composed of n substructures, its eigen-value problem of differential equation is as follows:

$$\boldsymbol{L}\boldsymbol{w}_k - \omega^2 \boldsymbol{M}\boldsymbol{w}_k = 0 \quad \text{in } \boldsymbol{\Omega} \tag{33}$$

$$\boldsymbol{B}\boldsymbol{w}_k = 0 \quad \text{on } \partial\boldsymbol{\Omega} \tag{34}$$

$$\boldsymbol{J}_j \boldsymbol{w}_k|_{s_j} = \overline{\boldsymbol{J}}_j \boldsymbol{w}_{k+1}|_{\bar{s}_j} + \overline{\boldsymbol{J}}_j \boldsymbol{w}_{k-1}|_{\bar{s}_j} \quad k = 1,2,\ldots,n \;\; j = 1,2,\ldots,l \tag{35}$$

where $\boldsymbol{w}_k$ denotes the generalized displacement vector of the kth substructure and $\boldsymbol{w}_0 \equiv \boldsymbol{w}_{n+1} = \boldsymbol{0}$. Eq. (35) represents the connections between two adjacent substructures. Considering the structure shown in Fig. 7, Eq. (35) is re-expressed as follows:

$$Q_k(s_1) + 2k_1 \sin^2 \alpha w_k(s_1) = k_1 \sin^2 \alpha [w_{k+1}(s_2) + w_{k-1}(s_2)] \tag{36}$$

$$Q_k(s_2) + 2k_1 \sin^2 \alpha w_k(s_2) = k_1 \sin^2 \alpha [w_{k+1}(s_1) + w_{k-1}(s_1)] \tag{37}$$

$$Q_k(s_3) + 2k_3 w_k(s_3) = k_3 w_{k+1}(s_3) + k_1 w_{k-1}(s_3) \tag{38}$$

$$Q_k(s_4) + 2k_4 w_k(s_4) = k_4 w_{k+1}(s_4) + k_4 w_{k-1}(s_4) \tag{39}$$

where α represents the angle between the spring and the beam. In Eq. (39), $k_4 \to \infty$ implies the rigid connection, i.e.

$$w_k(s_4) = 0, \quad k = 1,2,\ldots,n$$

5.2. Simplification of eigen-value problem and qualitative properties of modes

The modes of the structure shown in Fig. 6 have the following form:

$$\boldsymbol{w}^{(r)} = \left\{ w_1^{(r)} \quad w_2^{(r)} \quad \cdots \quad w_n^{(r)} \right\}^{\mathrm{T}} = \{ \sin r\psi \quad \sin 2r\psi \quad \cdots \quad \sin nr\psi \}^{\mathrm{T}} q_r \quad r = 1,2,\ldots,n \tag{40}$$

where $\psi = \pi/(n+1)$, and $\boldsymbol{w}_k^{(r)}$ denotes the displacement of the kth mass.

Based on the analysis in the preceding section and the idea of mode expansion, we can transfer the original displacement $\boldsymbol{w}$ into anther set of generalized displacements by a special transformation as follows:

$$\boldsymbol{w} = \begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \vdots \\ \boldsymbol{w}_n \end{Bmatrix} = \sqrt{\frac{2}{n+1}} \begin{bmatrix} \sin\psi\boldsymbol{I} & \cdots & \sin r\psi\boldsymbol{I} & \cdots & \sin n\psi\boldsymbol{I} \\ \sin 2\psi\boldsymbol{I} & \cdots & \sin 2r\psi\boldsymbol{I} & \cdots & \sin 2n\psi\boldsymbol{I} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sin n\psi\boldsymbol{I} & \cdots & \sin rn\psi\boldsymbol{I} & \cdots & \sin nn\psi\boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_r \\ \vdots \\ \boldsymbol{q}_n \end{Bmatrix} = \boldsymbol{Cq} \tag{41}$$

where the matrix $\boldsymbol{C}$ has following properties,

$$\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C} = \boldsymbol{I} \tag{42}$$

$$\boldsymbol{C}^{\mathrm{T}}(\boldsymbol{Y} + \boldsymbol{Y}^{n-1})\boldsymbol{C} = \mathrm{diag}(2\cos\psi\boldsymbol{I}, 2\cos 2\psi\boldsymbol{I}, \ldots, 2\cos n\psi\boldsymbol{I}) \tag{43}$$

where $\boldsymbol{Y}$ and $\boldsymbol{Y}^{n-1}$ are the matrices defined in Eq. (21). The result above can be obtained by applying the identical equation

$$\sin(k-1)r\psi + \sin(k+1)r\psi = 2\cos(r\psi)\sin(kr\psi) \tag{44}$$

Eqs. (33) and (34) are rewritten as follows:

$$\boldsymbol{L}'\boldsymbol{w} - \omega^2\boldsymbol{M}'\boldsymbol{w} = 0 \quad \text{in } \boldsymbol{\Omega} \tag{45}$$

$$\boldsymbol{B}'\boldsymbol{w} = 0 \quad \text{on } \partial\boldsymbol{\Omega} \tag{46}$$

$$\boldsymbol{J}'_j\boldsymbol{w}|_{s_j} = \overline{\boldsymbol{J}'_j}(\boldsymbol{Yw}|_{\bar{s}_j} + \boldsymbol{Y}^{n-1}\boldsymbol{w}|_{\bar{s}_j}) \quad j = 1, 2, \ldots, l \tag{47}$$

where $\boldsymbol{L}'$, $\boldsymbol{M}'$, $\boldsymbol{B}'$, $\boldsymbol{J}_j$ and $\overline{\boldsymbol{J}'}_j$ $(p = 1, 2, \ldots, n-1)$ are block diagonal matrices of $\boldsymbol{L}$, $\boldsymbol{M}$, $\boldsymbol{B}$, $\boldsymbol{J}_j$ and $\overline{\boldsymbol{J}}_j$, respectively. In these equations $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$ are coupled with each other.

Substituting the transformation equation (41) into Eqs. (45)–(47), then pre-multiplying with $\boldsymbol{C}^{\mathrm{T}}$, and using Eqs. (42) and (43), we may obtain,

$$\boldsymbol{L}\boldsymbol{q}_r - \omega^2\boldsymbol{M}\boldsymbol{q}_r = 0 \quad \text{in } \boldsymbol{\Omega} \tag{48}$$

$$\boldsymbol{B}\boldsymbol{q}_r = 0 \quad \text{on } \partial\boldsymbol{\Omega} \tag{49}$$

$$\boldsymbol{J}_j\boldsymbol{q}_r|_{s_j} = \overline{\boldsymbol{J}}_j 2\cos r\psi\boldsymbol{q}_r|_{\overline{s_j}} \quad r = 1, 2, \ldots, n \tag{50}$$

We draw two conclusions regarding to a chain structure:

(1) The eigen-value problem of the whole structure of a chain structure, as expressed in Eqs. (33)–(35), can be simplified into n eigen-value problems of a single substructure with different constraints, as expressed in Eqs. (48)–(50). Therefore, the mode of the whole structure can be obtained according to the following relationship,

$$\boldsymbol{w}_r = \{\boldsymbol{w}_{r1}, \boldsymbol{w}_{r2}, \ldots, \boldsymbol{w}_{rn}\}^{\mathrm{T}} = [\sin r\psi\boldsymbol{I}, \sin 2r\psi\boldsymbol{I}, \ldots, \sin nr\psi\boldsymbol{I}]^{\mathrm{T}}\boldsymbol{q}_r \quad r = 1, 2, \ldots, n \tag{51}$$

(2) The modes of a chain structure can be divided into n groups, each of which possesses the properties as expressed in Eq. (51).

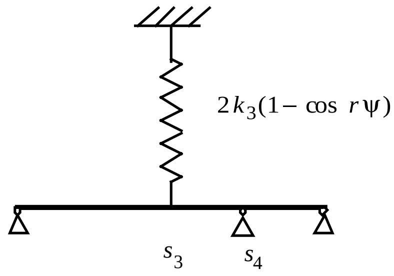

Fig. 8. The equivalent substructure.

5.3. Application

When we seek to solve the eigen-value problem of a chain structure using numeric method, we only need to solve n eigen-value problems of a single substructure, as expressed in (48)–(50). The mode of the whole structure can then be obtained according to expression (51). Therefore, the computational complexity can be considerably reduced.

If we want to obtain the modes and natural frequencies by experiment, we only need to measure n mode data $\boldsymbol{q}_r$ ($r = 1, 2, \ldots, n$) on the first substructure, and $\overline{\boldsymbol{q}}_r(s)$, the value of mode at some point s on the second substructure, where $\boldsymbol{q}_r(s)$ are not zero. Then we find out the value of r in the relationship $\overline{\boldsymbol{q}}_r(s) = (\sin 2r\psi / \sin r\psi)\boldsymbol{q}_r(s)$. As a result, the modes of the whole structure, $\boldsymbol{w}_r$, can be obtained by Eq. (51).

5.4. Examples

If the structure in Fig. 7 has only one spring connection on s_3 and a rigid connection on s_4, the constraints (50) are

$$\begin{aligned} Q_r(s_3) &= -2k_3(1 - \cos r\psi)q_r(s_3) \\ q_r(s_4) &= 0 \end{aligned} \tag{52}$$

Under these constraints, the substructure is equivalent to the following beam as illustrated in Fig. 8.

6. Axis-symmetric structures

6.1. Model and equation

A structure is termed axi-symmetric, if its geometry, physical properties, and boundary conditions are all unaltered after rotating it by an arbitrary angle with respect to a straight line—the axis. If this axis is set as the z-axis in a cylindrical coordinate system—$Or\theta z$, the geometry, physical properties and boundary conditions of an axi-symmetric structure are independent of θ.

In a three-dimensional continuous system, the eigen-value equation and boundary conditions of a axi-symmetric structure are expressed as follows:

$$\begin{aligned} &\boldsymbol{L}_{r,\theta,z}(r,z)[u(r,\theta,z), v(r,\theta,z), w(r,\theta,z)] - \omega^2 \boldsymbol{M}_{r,\theta,z}(r,z)[u(r,\theta,z), v(r,\theta,z), w(r,\theta,z)] = 0 \quad \text{in } \boldsymbol{\Omega} \\ &\boldsymbol{B}_{r,\theta,z}(r,z)[u(r,\theta,z), v(r,\theta,z), w(r,\theta,z)] = 0 \quad \text{on } \partial\boldsymbol{\Omega} \end{aligned} \tag{53}$$

where $\boldsymbol{\Omega}$ represents a three-dimensional domain in the cylindrical coordinate system $Or\theta z$. u, v, w denote the displacements in the direction of r, θ and z respectively, and $\boldsymbol{L}_{r,\theta,z}$, $\boldsymbol{M}_{r,\theta,z}$, $\boldsymbol{B}_{r,\theta,z}$ denote elastic, inertia and boundary condition differential operator matrices respectively. By virtue of the axi-symmetry, all the coefficients of these operator matrices are independent of θ.

For a two-dimensional problem (e.g. circular plane membrane, plate, and rotational shell), the coordinates are (r,θ) or (θ,z). For a one-dimensional problem (e.g. circular ring), the coordinate should only be θ. In addition, in some problems, only the displacement of u and v (in plane membrane problem) or w (in bending plate problem) appear in Eq. (53).

6.2. Properties of modes

In the following analysis of a axi-symmetric structure, the most complex case, a three-dimensional elastic body with displacements u, v, and w, is considered. Due to the axi-symmetry, the displacements of the structure possess the periodicity of 2π with respect to θ, which thus can be expanded into Fourier series of θ as follows:

$$\begin{aligned} u(r,\theta,z) &= \sum_{n=0}^{\infty}[U_n(r,z)\cos n\theta + U_n'(r,z)\sin n\theta] \\ v(r,\theta,z) &= \sum_{n=0}^{\infty}[V_n(r,z)\cos n\theta + V_n'(r,z)\sin n\theta] \\ w(r,\theta,z) &= \sum_{n=0}^{\infty}[W_n(r,z)\cos n\theta + W_n'(r,z)\sin n\theta] \end{aligned} \tag{54}$$

$\boldsymbol{L}_{r,\theta,z}$, $\boldsymbol{M}_{r,\theta,z}$, $\boldsymbol{B}_{r,\theta,z}$ are linear operator matrices and all of their coefficients are independent of θ. Substituting Eq. (54) into Eq. (53), due to the orthogonality of $\cos n\theta$, $\sin n\theta$, the harmonic wave of different orders is uncoupled. Therefore, the 3-dimensional eigen-value problem (53) can be simplified into a series of 2-dimensional ones:

$$\begin{aligned} &\boldsymbol{L}_{r,\theta,z}(r,z)[U_n\cos n\theta + U_n'\sin n\theta, V_n\cos n\theta + V_n'\sin n\theta, W_n\cos n\theta + W_n'\sin n\theta] \\ &\quad - \omega^2\boldsymbol{M}_{r,\theta,z}(r,z)[U_n\cos n\theta + U_n'\sin n\theta, V_n\cos n\theta + V_n'\sin n\theta, W_n\cos n\theta + W_n'\sin n\theta] = 0 \quad \text{in } \boldsymbol{\Omega} \\ &\boldsymbol{B}_{r,\theta,z}(r,z)[U_n\cos n\theta + U_n'\sin n\theta, V_n\cos n\theta + V_n'\sin n\theta, W_n\cos n\theta + W_n'\sin n\theta] = 0 \quad \text{on } \partial\boldsymbol{\Omega} \end{aligned} \tag{55}$$

In sequence, the modes of the structure have the form:

$$\boldsymbol{U}_n = \begin{bmatrix} u(r,\theta,z) \\ v(r,\theta,z) \\ w(r,\theta,z) \end{bmatrix} = \begin{bmatrix} U_n(r,z) \\ V_n(r,z) \\ W_n(r,z) \end{bmatrix}\cos n\theta + \begin{bmatrix} U_n'(r,z) \\ V_n'(r,z) \\ W_n'(r,z) \end{bmatrix}\sin n\theta \quad n = 0,1,2,\ldots \tag{56}$$

It should be noted that the structure is also symmetric with respect to any plane that contains the axis. If we use another cylindrical coordinates system $Or'\theta'z'$, where only the direction of θ' is converse with that in the original coordinates, and others are kept unaltered, the eigen-value problem expressed in the new coordinate system is

$$\begin{aligned} &\boldsymbol{L}'_{r',\theta',z'}(r',z')[u'(r',\theta',z'), v'(r',\theta',z'), w'(r',\theta',z')] - \omega^2\boldsymbol{M}'_{r,\theta,z}(r',z')[u'(r',\theta',z'), v'(r',\theta',z'), w'(r',\theta',z')] = 0 \\ &\quad \text{in } \boldsymbol{\Omega} \\ &\boldsymbol{B}'_{r',\theta',z'}(r',z')[u'(r',\theta',z'), v'(r',\theta',z'), w'(r',\theta',z')] = 0 \quad \text{on } \partial\boldsymbol{\Omega} \end{aligned} \tag{57}$$

where u', v', and w' represent the displacements and are the functions of r', θ', and z'. They have following relationship with u, v, w,

$$
\begin{aligned}
u'(r',\theta',z') &= u(r,-\theta,z)\\
v'(x',\theta',z') &= -v(r,-\theta,z)\\
w'(x',\theta',z') &= w(r,-\theta,z)
\end{aligned}
\tag{58}
$$

Because of the symmetric property mentioned above, the following relationships are satisfied,

$$
\boldsymbol{L}' = \boldsymbol{L}, \quad \boldsymbol{M}' = \boldsymbol{M}, \quad \boldsymbol{B}' = \boldsymbol{B}
\tag{59}
$$

Eqs. (58) and (59) indicate that

$$
\boldsymbol{U}_n^* = \begin{bmatrix} u(r,-\theta,z) \\ -v(r,-\theta,z) \\ w(r,-\theta,z \end{bmatrix} = \begin{bmatrix} U_n(r,z) \\ -V_n(r,z) \\ W_n(r,z) \end{bmatrix} \cos n\theta - \begin{bmatrix} U_n'(r,z) \\ -V_n'(r,z) \\ W_n'(r,z) \end{bmatrix} \sin n\theta
\tag{60}
$$

is also the eigen-vector of eigen-value problem (53), which is associated with the same eigen-value as eigen-vector expressed in (56) is.

Therefore, the linear combinations of $\boldsymbol{U}_n$ and $\boldsymbol{U}_n^*$

$$
\boldsymbol{U}_{n\mathrm{s}} = \frac{1}{2}[\boldsymbol{U}_n + \boldsymbol{U}_n^*] = \begin{bmatrix} U_n(r,z)\cos n\theta \\ V_n'(r,z)\sin n\theta \\ W_n(r,z)\cos n\theta \end{bmatrix}
\tag{61}
$$

$$
\boldsymbol{U}_{n\mathrm{a}} = \frac{1}{2}[\boldsymbol{U}_n - \boldsymbol{U}_n^*] = \begin{bmatrix} U_n'(r,z)\sin n\theta \\ V_n(r,z)\cos n\theta \\ W_n'(r,z)\sin n\theta \end{bmatrix}
\tag{62}
$$

are the modes corresponding to the same frequency. Eqs. (61) and (62) represent symmetric mode and anti-symmetric mode, respectively.

Moreover, because of the axi-symmetric nature of the structure, if we rotate the mode expressed in (61) by $\pi/2n$, we obtain

$$
\boldsymbol{U}_{n\mathrm{a}} = \begin{bmatrix} U_n(r,z)\sin n\theta \\ -V_n'(r,z)\cos n\theta \\ W_n(r,z)\sin n\theta \end{bmatrix}
\tag{63}
$$

Concerning an axi-symmetric structure, we come to two conclusions:

First, the modes of an axi-symmetric elastic body have the property of a harmonic wave in direction of θ. For each wave number n ($n = 0,1,2,\ldots$), there are two groups of modes, symmetric modes and anti-symmetric modes, as expressed in Eqs. (61) and (63) respectively, both of which correspond to the same frequencies. Moreover, the anti-symmetric mode with wave number n can be obtained by rotating the symmetric mode by $\pi/2n$.

Second, substituting Eq. (61) into Eq. (53) yields the governing equation of U_n, V_n', and W_n with parameter n:

$$
\begin{aligned}
&\boldsymbol{L}_{r,z,n}[U_n(r,z), V_n'(r,z), W_n(r,z)] - \omega^2 \boldsymbol{M}_{r,z,n}[U_n(r,z), V_n'(r,z), W_n(r,z)] = 0 \quad \text{in } \boldsymbol{\Omega}\\
&\boldsymbol{B}_{r,z,n}[U_n(r,z), V_n'(r,z), W_n(r,z)] = 0 \quad \text{on } \partial\boldsymbol{\Omega} \quad n = 1,2,\ldots
\end{aligned}
\tag{64}
$$

Thus, infinite number of eigen-value problems of two dimensions replaces the eigen-value problem of three dimensions.

For two-dimensional axi-symmetric structures such as a rotational shell and a circular plate, and one-dimensional axi-symmetric structure like a circular ring, their modes possess the special forms of Eqs. (61) and (63).

6.3. Application

In practice, we only need to acquire a limited number of modes of an axi-symmetric structure. Therefore, when we seek to solve the eigen-value problem of an axi-symmetric structure using numeric method, we can simplify Eq. (53) with a limited number of eigen-value problems as expressed in Eq. (64). The dimensions of the problem can thus be reduced by one, and the number of DOF required for solving a discrete problem will be considerably decreased.

If we want to obtain the modes and natural frequencies of an axi-symmetric by experiment, we only need to measure the data on a contour to the detected wave number, and the data on a certain plane containing the axis of a three-dimensional body (or on a generatrix of a two-dimensional structure, or on a point of one-dimensional structure). Then by applying the properties of the mode as expressed in Eqs. (61) and (63), we can know the modes of the whole structure.

7. Forced vibration problem for repetitive structures

For forced vibration problems of repetitive structures, the force vector can be transformed using the same method as applied to the generalized displacements mentioned in the previous sections. Therefore, the forced vibration problem of the whole structure is simplified into a group of uncoupled forced vibration problems of a single substructure.

For example, for a symmetric structure, its forced vibration equation is

$$\boldsymbol{L}\boldsymbol{w}_i + \boldsymbol{M}\ddot{\boldsymbol{w}}_i = \boldsymbol{F}_i \quad \text{in } \boldsymbol{\Omega} \quad i = 1, 2 \tag{65}$$

If the given force vector is transferred as follows:

$$\boldsymbol{F} = \begin{Bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \end{Bmatrix} = \boldsymbol{S}\boldsymbol{f} = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{Bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{f}_1 + \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \boldsymbol{f}_2 \tag{66}$$

Then in inverse,

$$\boldsymbol{f} = \begin{Bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \end{Bmatrix} = \boldsymbol{S}^{\mathrm{T}}\boldsymbol{F} = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \end{Bmatrix} \tag{67}$$

Substituting Eqs. (7) and (66) into Eq. (65) yields

$$\boldsymbol{L}\boldsymbol{q}_i + \boldsymbol{M}\ddot{\boldsymbol{q}}_i = \boldsymbol{f}_i \quad \text{in } \boldsymbol{\Omega} \quad i = 1, 2 \tag{68}$$

Similar method can be used in other kinds of repetitive structures.

8. Conclusions

In this paper, we discussed the free and forced vibrations of symmetric structures, cyclic periodic structures, linear periodic structures, chain structures, and axi-symmetric structures. The properties of modes for continuous models of repetitive structures are obtained by applying a series of transformation to the displacement function fields of these models. Compared to the research for discrete systems, the present discussion for continuous systems has more significance in theory.

According to these reduction approaches, the problem of calculating the natural and forced vibrations of the whole structure is simplified by calculating a group of relevant problems on a single substructure. Moreover, taking advantage of the specific properties of the modes, the vibration experiment can be simplified as well, which measurement only need to be performed on one substructure and on one point of an adjacent substructure. The effort of calculation and measurement is thus considerably reduced.

Utilizing the criterion that the data violating the qualitative properties are sure to be incorrect, the qualitative properties of the modes can be used to exam the correctness of the mode data obtained from calculation and experiment and the reasonableness of the design data given in inverse problem in vibration. In addition, they can be used to identify the reasonableness of the discrete models of the structures. For instance, the qualitative properties of modes for discrete models of symmetric, cyclic periodic, and chain structures, as derived by Chan et al. (1998) and Wang and Wang (2000) are the same as those for continuous models, if: (1) the displacement function fields, $\boldsymbol{w}$, $\boldsymbol{w}_r$, $\boldsymbol{q}$, $\boldsymbol{q}_r$, in Eqs. (7), (15) and (41), of both the whole structure and the substructures of a continuous model are replaced respectively with the generalized displacement vectors of discrete models, and (2) the dimensions of unit matrices $\boldsymbol{I}$ in these same equations are set respectively as the dimensions of generalized displacement vectors of substructures of corresponding discrete model. These facts show that the discrete models of these considered repetitive structures are reasonable in terms of their qualitative properties.

Acknowledgement

This work is supported by the National Natural Science Foundation of China (Grant no. 60034010).

References

Brillouin, L., 1946. Wave Propagation in Periodic Structures. Dover Publications, New York.

Cai, C., Wu, F., 1983. On the Vibration of Rotational Periodic Structures. In: Acta Scientiarum Naturalium Universitatis Sunyatseni, vol. 22, No. 3, p. 109.

Cai, C., Cheung, Y., Chan, H., 1990. Uncoupling of dynamic equations for periodic structures. Journal of Sound and Vibration 139 (2), 253–263.

Chan, H., Cai, C., Cheung, Y., 1998. Exact Analysis of Structures with Periodicity using U-Transformation. World Scientific Publication, Singapore.

Evensen, D.A., 1976. Vibration Analysis of Multi-symmetric structures. AIAA Journal 14 (4), 446–453.

Hu, H., Chen, D., 1988. Investigation on modal characteristics of cyclosymmetric structures. Chinese Journal of Applied Mechanics 5 (3), 1–8 (in Chinese).

Kirchhoff, G.R., 1879. Monatsberichte, Berlin, p. 815.

Mead, D.J., 1973. A general theory of harmonic wave propagation in linear periodic systems with multiple coupling. Journal of Sound and Vibration 27 (2), 235–260.

Mead, D.J., 1975a. Wave propagation and natural modes in periodic systems. I. Mono-coupled systems. Journal of Sound and Vibration 40 (1), 1–18.

Mead, D.J., 1975b. Wave propagation and natural modes in periodic systems. II. Multi-coupled systems, with and without damping. Journal of Sound and Vibration 40 (1), 19–39.

Mead, D.J.A.S., Bansal, 1978. Mono-coupled periodic systems with a single disorder: free wave propagation. Journal of Sound and Vibration 61 (4), 481–496.

Thomas, D.L., 1979. Dynamics of rotational periodic structures. International Journal of Numerical Methods in Engineering 14, 81–102.

Wang, D., Wang, C.C., 2000. Natural vibration of repetitive structures. Chinese Journal of Mechanics 16 (2), 85–95.

Zhong, W., 1991. The eigen-value problem of the chain of identical substructures and the expansion method solution lasted on the eigen-vectors. In: Acta Mechanica Sinica, vol. 23, No. 1, pp. 72–81.